Unit Overview

In this unit, students study trigonometric functions. They graph these functions and analyze their behaviors. They explore inverse trigonometric functions. Students also solve trigonometric equations.

Student Focus

Main Ideas for success in lessons 14-1 and 14-2.

→ Draw angles in standard position.
→ Find the initial side and terminal side of an angle.
→ Identify coterminal angles.
→ Measure angles in degrees and radians, and convert from one to the other.

Example

Lesson 14-1:

![Diagram of angles in standard position]

Terminal side

Initial side

![Diagram of angles in standard position on a unit circle]

Terminal side

Initial side
Make use of structure. For each angle in standard position, find two positive angles and two negative angles that are coterminal with the given angle.

a. \(60^\circ\)

\[420^\circ, 780^\circ, -300^\circ, -660^\circ\]

b. \(-100^\circ\)

\[260^\circ, 620^\circ, -460^\circ, -820^\circ\]

The circumference of a circle is equal to \(2\pi r\), where \(r\) is the radius of the circle. The length of an arc subtended by a central angle of \(n\) degrees is equal to \(\frac{n}{360} (2\pi r)\).

![Diagram of a circle with an arc subtended by a central angle](image)

Example A

A pet gerbil runs the length of an arc on a circular wheel with radius 12.5 cm which subtends a central angle of 40°. What length did the gerbil travel?

\(\frac{40}{360} (2\pi (12.5))\) Substitute the radius and angle measure.

8.7 cm Simplify.
Lesson 14-2:

Angles may be measured in radians as well as in degrees. One radian is the measure of a central angle which intersects an arc equal in length to the radii.

![Diagram of a circle with a central angle AOB]

\[ m\angle AOB = 1 \text{ radian} \]

**Example A**

A Ferris wheel with a radius of 45 feet rotates at a speed of 2.5 revolutions per minute (rpm). Find the angular velocity, in radians per minute, and the linear velocity, in miles per hour, of a point on the outer edge of the Ferris wheel.

Angular velocity is the ratio of radians rotated to time.

\[
2.5(2\pi) = 5\pi \\
\text{Find the number of radians in 2.5 revolutions.}
\]

\[
\frac{5\pi \text{ radians}}{1 \text{ minute}} \\
\text{Express angular velocity as a ratio.}
\]

Linear velocity is the ratio of feet traveled to time in hours.

\[
2.5(2\pi(45)) \approx 706.86 \text{ feet} \\
\text{Find the approximate distance traveled by multiplying the number of revolutions by the circumference.}
\]

\[
\frac{706.86 \text{ feet}}{1 \text{ minute}} \\
\text{Express linear velocity as a ratio.}
\]

\[
\frac{706.86 \text{ feet}}{1 \text{ minute}} \times \frac{1 \text{ mile}}{5280 \text{ feet}} \times \frac{60 \text{ minutes}}{1 \text{ hour}} \\
\text{Multiply to change the units.}
\]

8.03 miles per hour

Simplify.
Unit Overview

In this unit, students study trigonometric functions. They graph these functions and analyze their behaviors. They explore inverse trigonometric functions. Students also solve trigonometric equations.

Student Focus

Main Ideas for success in lessons 16-1 and 16-2.

→ Label angles and coordinates on the unit circle.
→ Define reciprocal trigonometric functions using the unit circle.
→ Evaluate all six trigonometric functions of angles in standard position.

Example

Lesson 16-1:

Math Tip

The trigonometric functions cos(θ) and sin(θ) are defined as follows:

\[ \cos \theta = \frac{x}{r} \quad \text{where} \quad x^2 + y^2 = r^2 \]

\[ \sin \theta = \frac{y}{r} \quad \text{where} \quad x^2 + y^2 = r^2 \]
In this figure, each of the four points, $P_1, P_2, P_3,$ and $P_4,$ can be described by an angle of measure $\theta,$ using different initial sides and rotation in different directions.

![Diagram of points and angles]

**Example A**

Given a $12^\circ$ angle in standard position that intersects a unit circle with point $P(0.978, 0.208)$ on the terminal side of a $12^\circ$ angle, the points of intersection for the terminal sides of three other angles can be found.

A $168^\circ$ angle intersects the unit circle at $P(-0.978, 0.208),$ because $168 = 180 - 12.$

A $192^\circ$ angle intersects the unit circle at $P(-0.978, -0.208),$ because $192 = 180 + 12.$

A $348^\circ$ angle intersects the unit circle at $P(0.978, -0.208),$ because $348 = 360 + (-12).$

Using the definitions of sine, $\sin \theta = \frac{y}{r},$ we see that $\sin 12^\circ = \sin 168^\circ,$ $\sin 12^\circ = -\sin 192^\circ,$ and $\sin 12^\circ = -\sin 348^\circ.$
Lesson 16-2:

The **reciprocal functions** of sine, cosine, and tangent are cosecant, secant, and cotangent, respectively.

**Example A**

Let $(4, -3)$ be a point on the terminal side of $\theta$, an angle in standard position. Find the values of sine, cosine, tangent, cosecant, secant, and cotangent of $\theta$.

We know that $x = 4$ and $y = -3$. So

$$r = \sqrt{x^2 + y^2} = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$$

Therefore, applying the definitions from Item 1, we know that

- $\sin \theta = -\frac{3}{5}$
- $\csc \theta = -\frac{5}{3}$
- $\cos \theta = \frac{4}{5}$
- $\sec \theta = \frac{5}{4}$
- $\tan \theta = -\frac{3}{4}$
- $\cot \theta = -\frac{4}{3}$

**Example B**

Given that $\sin \theta = -\frac{2}{5}$ and that $\cos \theta < 0$, find the values of the other five trigonometric functions of $\theta$. From the definition of sine, we know that $y = -2$ and $r = 5$. So $r^2 = x^2 + y^2 \Rightarrow 25 = x^2 + 4 \Rightarrow x = \pm \sqrt{21}$. Since $\cos \theta < 0$, we know that $x = -\sqrt{21}$. Therefore, applying the definitions from Item 1, we know that if

$$\sin \theta = -\frac{2}{5}$$

then

- $\csc \theta = -\frac{5}{2}$
- $\cos \theta = -\frac{\sqrt{21}}{5}$
- $\sec \theta = -\frac{5}{\sqrt{21}}$ or $-\frac{5\sqrt{21}}{21}$
- $\tan \theta = \frac{2}{\sqrt{21}}$ or $\frac{2\sqrt{21}}{21}$
- $\cot \theta = -\frac{\sqrt{21}}{2}$ or $\frac{\sqrt{21}}{2}$
Unit Overview

In this unit, students will build on their understanding of right triangle trigonometry as they study angles in radian measure, trigonometric functions, and circular functions. Students will investigate in depth the graphs of the sine and cosine functions and extend their knowledge of trigonometry to include tangent, cotangent, secant, and cosecant, as well as solving trigonometric equations.

Student Focus

Main Ideas for success in lessons 17-1 and 17-2:

→ Graph trigonometric functions over a given interval.
→ Describe how changes in the parameters affect the graphs.
→ Find the amplitudes and periods of trigonometric graphs, and write the function given a graph.

Example

Lesson 17-1:

Example A

Sketch the graphs of \( f(x) = \sin x \) and \( g(x) = 2 \sin x \) and describe how the two graphs differ.

Solution:

Graph the parent graph, \( y = f(x) \), labeling the scale of each axis. Use your knowledge of transformations, a calculator, or a table of values to graph \( y = g(x) \). Extend the graphs across the entire grid.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( \pi )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td>2( \pi )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Math Tip

Recall that the amplitude of a periodic function is defined to be one-half of the difference between the maximum and minimum function values.

Example

Lesson 17-2:

Example A

Write an equation for the graph below in terms of sine.
### Math Tip

For cosine, a horizontal shift $C$ can be determined by finding the distance a maximum (or minimum) point has been shifted from the $y$-axis.

<table>
<thead>
<tr>
<th>$y = A \sin B(x - C) + D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
</tr>
<tr>
<td>$D = \frac{\text{maximum} + \text{minimum}}{2}$</td>
</tr>
</tbody>
</table>

The period may be determined by finding the distance between two consecutive maximum values or two consecutive minimum values. Period $= 11\pi - 3\pi = 8\pi$

| $B = \frac{2\pi}{\text{period}}$ | $B = \frac{2\pi}{8\pi} = \frac{1}{4}$ |

A horizontal shift $C$ can be determined for sine by finding the $x$-coordinate of a point of intersection of the graph and the line $y = D$.

The graph intersects $y = -2$ at $(7\pi, -2)$, so a possible value of $C$ is $\pi$.

Determine whether $A$ is positive or negative by determining whether or not there is a vertical reflection.

The first extrema for $x > \pi$ is a maximum. Therefore, there is no vertical reflection, and so $A = 3$.

$$y = 3 \sin \left[ \frac{1}{4}(x - \pi) \right] - 2$$
Unit Overview

In this unit, students will build on their understanding of right triangle trigonometry as they study angles in radian measure, trigonometric functions, and circular functions. Students will investigate in depth the graphs of the sine and cosine functions and extend their knowledge of trigonometry to include tangent, cotangent, secant, and cosecant, as well as solving trigonometric equations.

Student Focus

Main Ideas for success in lesson 18-1:

→ Graph the reciprocal trigonometric functions, and determine the domain and range.
→ Find the period and locate asymptotes for the reciprocal trigonometric functions.

Example

Lesson 18-1:

<table>
<thead>
<tr>
<th>x</th>
<th>sin x</th>
<th>csc x</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>undefined</td>
</tr>
<tr>
<td>π/4</td>
<td>(\frac{\sqrt{2}}{2} \approx 0.707)</td>
<td>(\sqrt{2} \approx 1.414)</td>
</tr>
<tr>
<td>π/2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3π/4</td>
<td>(\frac{\sqrt{2}}{2} \approx 0.707)</td>
<td>(\sqrt{2} \approx 1.414)</td>
</tr>
<tr>
<td>π</td>
<td>0</td>
<td>undefined</td>
</tr>
<tr>
<td>5π/4</td>
<td>(\frac{\sqrt{2}}{2} \approx -0.707)</td>
<td>(\sqrt{2} \approx 1.414)</td>
</tr>
<tr>
<td>3π/2</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>7π/4</td>
<td>(\frac{\sqrt{2}}{2} \approx -0.707)</td>
<td>(\sqrt{2} \approx -1.414)</td>
</tr>
<tr>
<td>2π</td>
<td>0</td>
<td>undefined</td>
</tr>
</tbody>
</table>

Math Tip

The amplitude of a cosecant function is not defined because cosecant does not have a minimum and a maximum value.
Complete the following information for \( f(x) = \csc x \).

Period: \( 2\pi \)

Vertical asymptotes: \( x = k\pi \), where \( k \) is any integer

Domain: \( x \neq k\pi \), where \( k \) is any integer

Range: \( |y| \geq 1 \)

Zeros: \( f(x) = \csc x \) is undefined when \( x = 0 \).

Increasing: \( \ldots \cup (\frac{\pi}{2}, \pi) \cup (\pi, \frac{3\pi}{2}) \cup \ldots \)

Decreasing: \( \ldots \cup (0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi) \cup \ldots \)

**Graph of \( y = \csc(x) = 1/\sin(x) \)**

![Graph of csc(x)](https://example.com/graph.png)
Unit Overview

In this unit, students will build on their understanding of right triangle trigonometry as they study angles in radian measure, trigonometric functions, and circular functions. Students will investigate in depth the graphs of the sine and cosine functions and extend their knowledge of trigonometry to include tangent, cotangent, secant, and cosecant, as well as solving trigonometric equations.

Student Focus

Main Ideas for success in lesson 19-1, 19-2, and 19-3:

→ Define and apply inverse trigonometric functions to real-world situations.
→ Find values of inverse trigonometric functions.

Example

Lesson 19-1:

Because the function \( y = \cos(x) \) is one-to-one over the domain \( 0 \leq x \leq \pi \), the inverse of the function over this domain is a function and an example of an inverse trigonometric function. Although there are other domains over which the cosine function is one-to-one, this is the function that mathematicians have chosen to use to define the inverse cosine function. The notation for the inverse cosine function is \( y = \cos^{-1}(x) \), where \( \cos(y) = x \) and \( 0 \leq y \leq \pi \).

To find the values for \( y = \cos^{-1}(x) \), interchange the values of \( y = \cos(x) \) so that the x-values become the y-values, and the y-values become the x-values.
Example

Lesson 19-2:

Because the function \( y = \sin(x) \) is one-to-one over the domain \( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \), the inverse of the function over this domain is a function. Although there are other domains over which the sine function is one-to-one, this is the function that mathematicians have chosen to use to define the inverse sine function. The notation for the inverse sine function is \( y = \sin^{-1}(x) \), where \( \sin(y) = x \), and \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\).

To find the values for \( y = \sin^{-1}(x) \), interchange the values of \( y = \sin(x) \) so that the x-values become the y-values, and the y-values become the x-values.
Example

Lesson 19-3:

In a similar manner, because the function $y = \tan(x)$ is one-to-one over the domain $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and although there are other domains over which the tangent function is one-to-one, this is the function that mathematicians have chosen to use to define the inverse tangent function. The notation for the inverse tangent function is $y = \tan^{-1}(x)$, where $\tan(y) = x$, and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

To find the values for $y = \tan(x)$, interchange the values of $y = \tan(x)$ so that the $x$-values become the $y$-values, and the $y$-values become the $x$-values.

---

**Math Tip**

The inverses of the reciprocal trigonometric functions can be defined as follows:

- $y = \csc^{-1}(x)$, where $\csc(y) = x$, and $-\frac{\pi}{2} < y \leq \frac{\pi}{2}, y \neq 0$.

- $y = \sec^{-1}(x)$, where $\sec(y) = x$, and $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$.

- $y = \cot^{-1}(x)$, where $\cot(y) = x$, and $0 < y < \pi$. 

---
Unit Overview

In this unit, students will build on their understanding of right triangle trigonometry as they study angles in radian measure, trigonometric functions, and circular functions. Students will investigate in depth the graphs of the sine and cosine functions and extend their knowledge of trigonometry to include tangent, cotangent, secant, and cosecant, as well as solving trigonometric equations.

Student Focus

Main Ideas for success in lessons 20-1 and 20-2:

→ Use inverse functions and reference angles to solve trigonometric equations.
→ Determine when solutions are limited to a given interval.

Example

Lesson 20-1:

**Example A**

Find the general solutions of \( \cos x = \frac{\sqrt{3}}{2} \).

Since the period of \( \cos x \) is \( 2\pi \), first find the solutions for \( \cos x = \frac{\sqrt{3}}{2} \) over the interval \([0, 2\pi]\). To do this, either visualize the graph above or visualize the unit circle.

<table>
<thead>
<tr>
<th>( \cos x = \frac{\sqrt{3}}{2} )</th>
<th>( x = \frac{\pi}{4} ) and ( x = \frac{7\pi}{4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Add integral multiples of ( 2\pi ) to find the general solutions.</td>
<td>( x = \frac{\pi}{4} + 2\pi k ) and ( x = \frac{7\pi}{4} + 2\pi k ), where ( k ) is any integer.</td>
</tr>
</tbody>
</table>
Example B

Solve \(4 \cos x + 9 = 11\) over the interval \([0, 2\pi]\). Compare this to solving the equation \(4x + 9 = 11\).

<table>
<thead>
<tr>
<th>Trigonometric equation</th>
<th>Corresponding algebraic equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4 \cos x + 9 = 11)</td>
<td>(4x + 9 = 11)</td>
</tr>
<tr>
<td>(4 \cos x = 2)</td>
<td>(4x = 2)</td>
</tr>
<tr>
<td>(\cos x = \frac{1}{2})</td>
<td>(x = \frac{1}{2})</td>
</tr>
</tbody>
</table>

The solutions to the equation \(\cos x = \frac{1}{2}\) over the interval \([0, 2\pi]\) are \(x = \frac{\pi}{3}\) and \(x = \frac{5\pi}{3}\).

**MATH TIP**

To find the general solutions to a trigonometric equation over the interval \((-\infty, \infty)\), find the solutions over one period of the trigonometric function, and then add integral multiples of the period.

**MATH TIP**

Since secant and cosine are reciprocal functions, the equations \(\sec \theta = a\) and \(\cos \theta = \frac{1}{a}\) will have the same solutions.

Since cosecant and sine are reciprocal functions, the equations \(\csc \theta = b\) and \(\sin \theta = \frac{1}{b}\) will have the same solutions.
Example

Lesson 20-2:

Reference angles can be useful when solving some trigonometric equations. If \( \theta \) is an angle in standard position, then its *reference angle* \( \alpha \) is the acute angle formed by the terminal side of \( \theta \) and the horizontal axis.
### Example A

Solve \(3 \cos \theta + 4 = 5\) over the interval \([0^\circ, 360^\circ]\). Give answers to the nearest tenth of a degree.

<table>
<thead>
<tr>
<th>Solve for (\cos \theta).</th>
<th>(3 \cos \theta + 4 = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Note that since (\cos \theta) is positive, solutions lie in Quadrants I and IV.</td>
<td>(\cos \theta = \frac{1}{3})</td>
</tr>
</tbody>
</table>

Find the reference angle for the solutions. Be sure your calculator is in degree mode.

\[
\alpha = \cos^{-1} \frac{1}{3} \approx 70.5^\circ
\]

The reference angle is the Quadrant I solution. Find the Quadrant IV solution by using \(\theta = 360^\circ - \alpha\).

\[
\alpha = 70.5^\circ, \ \theta = 289.5^\circ
\]

### Example B

Solve \(\sin x + 3.2 = 3\) over the interval \([0, 2\pi]\). Give answers to the nearest thousandth of a radian.

<table>
<thead>
<tr>
<th>Solve for (\sin x).</th>
<th>(\sin x + 3.2 = 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Note that since (\sin x) is negative, solutions lie in Quadrants III and IV.</td>
<td>(\sin x = -0.2)</td>
</tr>
</tbody>
</table>

When finding the reference angle for the solutions, ignore the negative sign. Be sure that your calculator is in radian mode.

\[
\alpha = \sin^{-1} 0.2 \\
\approx 0.201
\]

Find the Quadrant III solution by using \(x = \pi + \alpha\). Find the Quadrant IV solution by using \(x = 2\pi - \alpha\).

\[
x = 3.343 \\
x = 6.082
\]
**LESSON 14-1**

1. An angle of $99^\circ$ is drawn on a coordinate plane with its vertex at the origin and its initial side on the positive $x$-axis. In which quadrant does the terminal side lie?

   A. Quadrant I
   B. Quadrant II
   C. Quadrant III
   D. Quadrant IV

2. Find an angle between $0^\circ$ and $360^\circ$ that is coterminal with the given angle.
   
   a. $672^\circ$
   
   b. $-194^\circ$

3. **Model with mathematics.** The second hand of a clock is 6 inches long.

   a. How many degrees does the second hand rotate in 23 minutes?

   b. How far does the tip of the second hand travel in 23 minutes?

4. **Make sense of problems.** Find the perimeter of the given figure.

   ![Diagram of a triangle with a 36° angle and a side of 7 inches.]

5. A end-pivot irrigation pipe that is 113 feet long runs with an arc length of 65 feet. How many degrees does the pipe pivot?

**LESSON 14-2**

6. An angle of $\frac{15\pi}{13}$ radians is drawn on a coordinate plane with its vertex at the origin and its initial side on the positive $x$-axis. In which quadrant does the terminal side lie?

   A. Quadrant I
   B. Quadrant II
   C. Quadrant III
   D. Quadrant IV

7. Find an angle between $0$ and $2\pi$ that is coterminal with the given angle.

   a. $\frac{3\pi}{5}$

   b. $\frac{61\pi}{7}$

8. **Model with mathematics.** The face on the clock of Big Ben has a diameter of 7 meters. What is the maximum linear speed of the tip of the second hand in meters per hour?

9. **Make sense of problems.** Explain why the answer in Item 8 is a maximum.
10. Find the coordinates of $P$ to three decimal places.

11. Let $t = 0$ seconds represent the time when Dexter’s front wheel starts to move. Sketch a graph of the height above the pavement of the paint spot on the front wheel as a function of the number of seconds for the first 8 seconds after $t = 0$.

12. What distance will the front wheel travel in 1.5 seconds?
   - A. $22\pi$ inches
   - B. $11\pi$ inches
   - C. $5.5\pi$ inches
   - D. $2.75\pi$ inches

13. Make use of structure. How long does it take for the back wheel to make one complete revolution?

14. Dexter draws a mark on the front tire with chalk. Sketch a figure that shows this mark at $145^\circ$ from the ground. How many seconds after the chalk mark touches the ground will the mark first be in this position?

15. What will be the approximate height of the chalk mark at the time stated in Item 14?

16. Given the graph of $f(x)$ below, which of the following is the period of the graph?

   A. 1
   B. 2.5
   C. 5
   D. 10
17. Which of the following is the amplitude of the graph of $f(x)$?

A. 1  
B. 2.5  
C. 5  
D. 10

18. Use appropriate tools strategically. Use a graphing calculator to find a sine or cosine function that best matches the periodic graph in Item 16.

19. Make use of structure. Let $g(x)$ be a vertical shift down 0.5 of $f(x)$ from Item 16. Graph $g(x)$.

20. Do the period and amplitude change for $g(x)$ with the shift? Explain.

LESSON 15-3

Model with mathematics. Suny’s unicycle has a front wheel with a 36-inch diameter. She is riding at a steady pace, and the wheel rotates once every 2 seconds. She also made a mark the front tire with chalk. Suppose that the height of the chalk mark is measured as a vertical distance above or below the center of the wheel. The mark starts at a point on the same horizontal line as the center of the wheel at $t = 0$. The wheel turns in the direction shown by the arrow in the figure at the given rate.

21. Draw a graph of the height of the mark as a function of time for $0 \leq t \leq 8$.

22. Make use of structure. Instead of defining the function as height versus time, consider defining it as the height of the chalk mark in feet versus the angle of rotation, measured in degrees, of spoke $s$. Through how many degrees will the spoke rotate in 3 seconds?
23. Redraw the graph from Item 21. Label the axes so that the graph illustrates the height of the chalk mark, in feet, as a function of the angle of rotation of spoke $s$, measured in radians.

![Graph of a function](image)

24. Use a graphing calculator to graph the function $y = \sin(x)$. What is the period and amplitude of the graph of $y = \sin(x)$?

25. How does the graph of the function in Item 23, including period and amplitude, compare to the graph of $y = \sin(x)$?

**LESSON 16-1**

Attend to precision. For Items 26–28, use the unit circle to give the exact value.

26. $\sin\left(\frac{3\pi}{4}\right)$

27. $\cos\left(\frac{\pi}{12}\right)$

28. $\tan 75^\circ$

29. Given $\tan\left(\frac{\pi}{7}\right) = 0.482$, find $\tan\left(\frac{8\pi}{7}\right)$.

30. Make use of structure. Given $\sin 85^\circ = 0.848$, explain how you can use the Pythagorean Theorem to find the coordinates of the point representing $85^\circ$ on the unit circle.

**LESSON 16-2**

Attend to precision. For Items 31–33, use the unit circle and the definitions of the reciprocal trigonometric functions to give the exact value.

31. $\cot\left(\frac{4\pi}{3}\right)$

32. $\csc 270^\circ$

33. $\sec \pi$

Make use of structure. For Items 34–35, find the values of the six trigonometric functions of $\theta$.

34. Point $P (-1, 5)$ on the terminal side of $\theta$, an angle in standard position

35. $\csc \theta = \frac{7}{3}$

**LESSON 17-1**

36. Attend to precision. Complete the table using the unit circle.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x) = \cos 3x$</th>
<th>$h(x) = 3 \cos x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{3\pi}{4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{5\pi}{4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{7\pi}{4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2\pi$</td>
<td></td>
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</tr>
</tbody>
</table>

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37. Use the axes below to graph the functions in Item 36 from $0 \leq x \leq 4\pi$.

[Graph of a function]

38. What is the period of $g(x)$?
   A. $2\pi$
   B. $\frac{3\pi}{2}$
   C. $\frac{2\pi}{3}$
   D. $3\pi$

39. If $f(x)$ is the parent function, what are the period and amplitude of $f(x)$, $g(x)$, and $h(x)$?

40. Based on your work in Items 36–39, what can you conclude about $A$ and $B$ in the equation $y = A \cos Bx$?

**LESSON 17-2**

Make use of structure. For Items 41–43, state the period and amplitude of each function and describe any phase shifts. Sketch the graph of each function over one period. Carefully label the scale on each axis.

41. $y = 2.5 \cos x$

[Graph of $y = 2.5 \cos x$]

42. $y = \sin \left(\frac{x}{3}\right)$

[Graph of $y = \sin \left(\frac{x}{3}\right)$]
43. \( y = \frac{1}{2} \sin \left( x - \frac{\pi}{4} \right) \)

44. For the function \( y = -2 \cos(5x - 2\pi) + \frac{3}{4} \), state the period and amplitude. Describe any horizontal or vertical shifts relative to the parent graph.

45. **Construct viable arguments.** Explain why the function in Item 44 is not a shift of the parent function to the right by \( 2\pi \).

**LESSON 18-1**

46. Which of the following is the reciprocal function of the cosine function?
   A. cosecant
   B. secant
   C. tangent
   D. cotangent

47. **Attend to precision.** Complete the table of values using the unit circle.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = \cos x )</th>
<th>( g(x) = \sec x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
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<tr>
<td>( \frac{\pi}{4} )</td>
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<tr>
<td>( 2\pi )</td>
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</tr>
</tbody>
</table>

48. Sketch the graph of \( f(x) \) and \( g(x) \) from the values found in Item 47.

49. **Reason abstractly.** Explain how to locate vertical asymptotes of the graph of \( g(x) \) in Item 47.
50. Describe the period, vertical asymptotes, domain, range, and zeros for \( g(x) \) in Item 47.

**LESSON 18-2**

**Make use of structure.** For Items 51–53, state the period and amplitude of each function and describe any phase shifts. Sketch the graph of each function over one period. Carefully label the scale on each axis.

51. \( y = \frac{3}{4} \sec x \)

52. \( y = \csc \pi x \)

53. \( y = 2 \cot \left( x + \frac{\pi}{6} \right) \)

54. Describe the transformation of \( y = \sec x \) to \( y = \sec(x - 3\pi) \) in words.

55. Describe the transformation of \( y = \tan x \) to \( y = -\tan(x + 2) \) in words.

**LESSON 19-1**

56. **Attend to precision.** Find the exact value of each expression using the unit circle.

   a. \( \cos^{-1} \left( \frac{-\sqrt{3}}{2} \right) \)

   b. \( \cos^{-1} \left( \frac{1}{2} \right) \)

57. **Use appropriate tools strategically.** Find the approximate value for each expression using a calculator in radian measure, correct to three decimal places.

   a. \( \cos^{-1}(-0.48) \)

   b. \( \cos^{-1}(1.32) \)

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58. The adjacent side of angle $\theta$ in a right triangle has a length of 6. The hypotenuse has a length of 11. Which of the following is the correct equation for finding the measure of angle $\theta$?

A. $\cos \left( \frac{6}{11} \right) = \theta$

B. $\cos \theta = \frac{6}{11}$

C. $\cos^{-1} \left( \frac{6}{11} \right) = \theta$

D. $\cos^{-1} \theta = \frac{6}{11}$

59. Regina is looking at the top of the flagpole. The line-of-sight distance between Regina and the top of the flagpole is 42 feet. If the horizontal distance between Regina and the flagpole is 23 feet, calculate the approximate angle of elevation (in degrees).

60. Kalem says that the reciprocal and inverse function of $y = \cos x$ are the same. Is he correct? If not, explain why they are not the same.

61. Attend to precision. Find the exact value of each expression using the unit circle.

a. $\sin^{-1} \left( \frac{\sqrt{3}}{2} \right)$

b. $\sin^{-1} \left( -\frac{1}{\sqrt{2}} \right)$

62. Use appropriate tools strategically. Find the approximate value for each expression using a calculator in radian measure, correct to three decimal places.

a. $\sin^{-1}(0.957)$

b. $\sin^{-1}(2.01)$

63. The opposite side of angle $\theta$ in a right triangle has a length of 4. The hypotenuse has a length of 5. Which of the following is the correct equation for finding the measure of angle $\theta$?

A. $\sin^{-1} \left( \frac{4}{5} \right) = \theta$

B. $\sin^{-1} \theta = \frac{4}{5}$

C. $\sin^{-1} \left( \frac{5}{4} \right) = \theta$

D. $\sin \theta = \frac{5}{4}$

64. Reason quantitatively. Explain why $\sin^{-1} 4$ displays an error when you use a calculator to find its value.

65. A pole of a volleyball net is tethered to the ground with a rope that is 12 feet long. If the length of the pole is 6 feet, what is the measure, in degrees, of the angle of elevation $\theta$?
LESSON 19-3

66. **Attend to precision.** Find the exact value of each expression using the unit circle.
   a. \( \tan^{-1}(-\sqrt{3}) \)
   b. \( \tan^{-1}(-1) \)

67. **Use appropriate tools strategically.** Find the approximate value for each expression using a calculator in radian measure, correct to three decimal places.
   a. \( \tan^{-1}(1.795) \)
   b. \( \tan^{-1}(-0.74) \)

68. Without using a calculator, find the exact value of \( \cos^{-1}\left(\cos\left(\frac{\pi}{4}\right)\right) \).
   A. 0.707
   B. 0.667
   C. \( \frac{4}{\pi} \)
   D. \( \frac{\pi}{4} \)

69. **Reason abstractly and quantitatively.** Explain how you found your answer to Item 68.

70. Dion is at the top of a lighthouse looking at a boat that is 600 meters from the lighthouse. Calculate the approximate angle of elevation if the height of the lighthouse is 85 meters.

![Triangle Diagram]

LESSON 20-1

71. **Reason abstractly.** Find the general solutions of each expression.
   a. \( 2\sin x = \sqrt{2} \)
   b. \( 4\cos^2 x - 3 = 0 \)

72. **Express regularity in repeated reasoning.** Explain why \( 2k\pi \) is part of the solution to the functions in Item 71.

73. Find the exact solutions of each equation over the interval \([0, 2\pi]\).
   a. \( \csc^2 x - 1 = 3 \)
   b. \( 2\tan x = -2 \)
74. Find the exact solutions of each equation over the interval \([0^\circ, 360^\circ]\).
   \[ \text{a. } 2 \cos \theta = \sqrt{3} \]
   \[ \text{b. } \tan^3 \theta = -\sqrt{3} \tan^2 \theta \]

75. Which are the exact solutions of \(4 \sec \theta = 8\) over the interval \([0^\circ, 360^\circ]\)?
   \[ \text{A. } 45^\circ, 225^\circ \]
   \[ \text{B. } 45^\circ, 135^\circ \]
   \[ \text{C. } 30^\circ, 150^\circ \]
   \[ \text{D. } 30^\circ, 330^\circ \]

**LESSON 20-2**

**Use appropriate tools strategically.** For Items 76–77, find the solutions of each equation over the interval \([0, 2\pi]\). Give answers to the nearest tenth of a radian.

76. \(4 \csc x + 11 = 0\)

77. \(\sin x + 4 = 4.6\)

**Make use of structure.** For Items 78–79, find the solutions of each equation over the interval \([0^\circ, 360^\circ]\). Give answers to the nearest tenth of a degree.

78. \(8 \sin \theta = 3\)

79. \(\cos^2 \theta = \frac{9}{16}\)

80. Suppose \(\theta\) lies in Quadrant II and its reference angle is \(\alpha = 41.8^\circ\). What is the measure of \(\theta\)?
   \[ \text{A. } 41.8^\circ \]
   \[ \text{B. } 138.2^\circ \]
   \[ \text{C. } 221.8^\circ \]
   \[ \text{D. } 318.2^\circ \]